Quartic Polynomials With Three Relative Extrema and Two Points of Inflection at Integer Lattice Points

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Introduction The problem we consider is to find all polynomials with integer coefficients possessing three relative extrema and two points of inflection at integer lattice points. The proof breaks into four relatively easy steps. First we simplify the problem a little, without any loss of generality, then reduce it to solving a diophantine equation. Next we solve that equation with the help of a new arithmetic function, then make sure that our solutions work for the original problem. A high school student with a reasonable background in math should have little trouble understanding every step of the argument.

The main result, Theorem 1, can be used by calculus and physics teachers to generate problems whose solutions are always integers. Problems about points moving along curves, for example, are usually more agreeable if the velocity and acceleration are only zero when an integral length of time has elapsed. It is also much easier to sketch a curve if its critical points and points of inflection are at lattice points. A polynomial is called *nice* if it has integral coefficients, has integral roots, all of its derivatives have integral roots, and all of these roots are distinct. As a bonus, we get a classification of all nice cubic polynomials as a corollary. The topic of writing good calculus problems by finding nice cubic polynomials is discussed in [1] and [2]. A closely related problem is also solved in [4].

Simplification Consider the polynomial $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E \in \mathbb{Z}[x]$. Graphically, the constant E moves the curve up and down and thus has no effect on relative extrema and points of inflection, so we can let $E = 0$ without loss of generality. By replacing x with $x + r$ we get a new polynomial in $\mathbb{Z}[x]$ which is just the same curve shifted r units in the negative x direction. What this means is that given any polynomial in $\mathbb{Z}[x]$ with the five desired lattice points, we can use E and r to construct a new polynomial in $\mathbb{Z}[x]$ with the five desired lattice points, one of which is the origin. This process is reversible so we may assume without loss of generality that any one of the five lattice points is the origin. What it means for a quartic polynomial to have three relative extrema and two points of inflection at lattice points is that the first derivative has three integral roots and the second derivative has two, all five of which are distinct. If a polynomial is multiplied by a constant, then the roots of its derivatives are not altered, hence we can assume without loss of generality that A is positive (otherwise we multiply through by -1) and that A, B, C, and D share no common factor greater than 1 (otherwise we multiply though by its multiplicative inverse). Consider the first derivative $f'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D = 4A(x + a)(x + b)(x + c)$, where $-a$, $-b$, and $-c$ are the roots. Consider also the second derivative$f''(x) = 12Ax^2 + 6Bx + 2C$ which has the same roots as $6Ax^2 + 3Bx + C$. Call the roots m and m', with $m > m'$, so $6Ax^2 + 3Bx + C = 6A(x - m)(x - m')$. When we equate coefficients, we find that $B = (4A/3)(a + b + c) = -A(m + m')/2$, $C = 2A(ab + ac + bc) = 6Amm'$, and $D = 4Aabc$.

Therefore A divides A, B, C, and D, so without loss of generality $A = 1$. Thus, by the above equations and the quadratic formula, we have following equations, where the $\pm$ in the first one is $+$ for m and $-$ for m'.

$$m, m' = \frac{-3B \pm \sqrt{9B^2 - 24C}}{12}. \tag{1}$$

$$B = (4/3)(a + b + c). \tag{2}$$

$$C = 2(ab + ac + bc). \tag{3}$$

The Diophantine Equation The idea here is to find $a+b$ and ab in terms of B, m, and c, then use the quadratic formula to find a and b. First we must get C in terms of m and B, so we isolate C in Equation (1).

$$C = -6m^2 - 3Bm. \tag{4}$$

Since $a, b, c \in \mathbb{Z}$, from Equation (2) we see that $4|B$. Let $y = B/4$, then $a+b = 3y-c$. Combining Equations (3) and (4) with our equation for $a+b$ yields $-6m^2 - 3Bm = 2ab + 2c(a+b) = 2ab + 2c(3y-c)$, hence $ab = c^2 - 3cy + 6my - 3m^2$.

Now, a and b are the roots of $x^2 - (a+b)x + ab$ (let a be the larger root) and can only be integers when the discriminant $(a+b)^2 - 4ab$ is a perfect square, say w^2 (w is non-negative). Combining this w with our equations for $a+b$ and ab yields $w^2 = 9y^2 + 6cy + 24my - 3c^2 + 12m^2$.

At this point we use the fact that any one of the five roots can be set equal to 0 without loss of generality to let $m = 0$ so that $w^2 = 9y^2 + 6cy - 3c^2 = 3(y+c)(3y-c)$. Let $e = y+c$ so that $3y - c = 4y - e$; thus our equation for w becomes much simpler.

$$w^2 = 3e(4y-e). \tag{5}$$

Note that any three integers e, w, and y that satisfy Equation (5) can be used to construct a polynomial of the desired form as long as the five roots are distinct (this restriction will be checked later) and w is congruent to $3y-c$ modulo 2 (so that a and b are integers), and that any polynomial with the five desired lattice points must yield a solution to Equation (5).

Solution To solve Equation (5) we need a new arithmetic function. Define $\upsilon(x)$ to be the smallest positive integer whose square is divisible by x. If $e = \prod p_n^{e(n)}$, then $\upsilon(e) = \prod p_n^{\lceil e(n)/2 \rceil}$, where the products are over all primes. Thus $\upsilon(3e)|w$ since any perfect square which is divisible by x must also be divisible by $\upsilon(x)^2$. We now split the problem into two cases.

Case 1: $4|e$

Then there exists an integer k such that $4k = e$, thus $w^2 = 12e(y-k)$, so $\upsilon(12e)|w$ (note that $\upsilon(12e) = 2\upsilon(3e)$, which will be used later). If $w = \upsilon(12e)$, then $w^2/(12e)$ is an integer, so letting $y = w^2/(12e) + k$ yields an integral solution to Equation (5). If $w = t\upsilon(12e)$, then letting $y = \frac{w^2/(3e)+e}{4}$ yields an integral solution since $t^2\upsilon(12e)^2/(12e)$ and $e/4$ are both integers. Therefore Equation (5) is soluble when w is any integral multiple of $\upsilon(12e)$ (i.e. an even multiple of $\upsilon(3e)$).

Case 2: otherwise

From Equation (5) we see that $u(3e)|w$. If $w = \upsilon(3e)$, then $w^2/3e$ is an integer, so letting $y = \frac{w^2/(3e)+e}{4}$ yields a solution to Equation (5), but this time it's not as obvious that what we get is an integer. If e is even but not a multiple of 4, then $w^2/(3e)$ is also even but not a multiple of 4, so the numerator factors as 2 times the sum of two odd numbers, and hence is a multiple of 4. If e is odd, then $w^2/(3e)$ is also odd and is the product of primes which divide $3e$ an odd number of times. Therefore $w^2/(3e) \equiv 3e \pmod 4$. This means that $w^2/(3e)$ and e are not congruent modulo 4 since one of them is 1 and the other is 3, thus $\frac{w^2/(3e)+e}{4}$ is an integer. If $w = t\upsilon(3e)$, then $y = \frac{w^2/(3e)+e}{4}$ is an integer iff t is odd (since we need $t^2 \equiv 1 \pmod 4$), thus there is only a solution when w is an odd multiple of $\upsilon(3e)$.

Combining the cases, we find that Equation (5) has an integral solution if and only if

$$w = t\upsilon(3e), \tag{6}$$

where $t > 0$ is even if $4|e$, and odd otherwise.

Working Backwards Now that we have solved the diophantine equation that came from the original problem, we must check that no extraneous solutions were picked up along the way. This amounts to verifying that a and b are integers and that m, m', $-a$, $-b$, and $-c$ are all distinct. By the quadratic formula, we know that a and b are $(3y-c\pm w)/2$ (where the $\pm$ is $+$ for a and $-$ for b), which is an integer iff $w \equiv 3y-c \pmod 2$. We already know that $3y-c = 4y-e \equiv e \pmod 2$, but $w \equiv e \pmod 2$ by Equation (6), so $w \equiv 3y-c \pmod 2$, so a and b are always integers. To show the roots are distinct, we just consider all 10 cases.
Case 1: $-a=-b$
This occurs iff $w=0$, so we must require $w \neq 0$.
Case 2: $-a=-c$
Since $a=(3y-c+w)/2$, this case would make $c=(3y-c+w)/2$, or $w=3c-3y=3e-6y=3e-6(\frac{w^2/(3e)+e}{4})$, thus $w^2+(2e)w-3e^2=0$. The roots of this quadratic are $3e$ and $-e$, so we must require that w does not equal either one of these.
Case 3: $-b=-c$
Similarly (since $b=(3y-c-w)/2$), we get $w^2-(2e)w-3e^2=0$, which has roots $-3e$ and e, so we must require that w is neither one of these.
Case 4: $m=m'$
Since $m=0$, by Equation (1) we would need $B^2-8C=0$ and $B=0$. These imply $C=0$, so $f'(x)=4x^3+D$ which has only one real root. Therefore w is not real, which contradicts Equation (6).
Case 5: $-a=m$
Since $m=0$ and $-a=-(4y-e+w)/2$, we need $4y-e+w=0$. Hence $w=e-4(\frac{w^2/(3e)+e}{4})$, so $w=-w^2/(3e)$. Therefore $w=0$ or $w=-3e$, which we dealt with in Cases 1 and 3.
Case 6: $-b=m$
Similarly we get $w=3e$, which we dealt with in Case 2.
Case 7: $-a=m'$
By Equation (1) $m'=m-\frac{\sqrt{9B^2-24C}}{6}$, but $m=0$, so $C=0$ by Equation (4). Hence $m'=-3B/6=-2y$. If $m'=-a$, then $-2y=-(4y-e+w)/2$, so $w=e$, which we dealt with in Case 3.
Case 8: $-b=m'$
Similarly we get $w=-e$, which we dealt with in Case 2.
Case 9: $-c=m$
If $c=0$, then $e=y=\frac{w^2/(3e)+e}{4}$. Thus $w^2=9e^2$, so $w=\pm 3e$, which we dealt with in Cases 2 and 3.
Case 10: $-c=m'$
If $-c=-2y$, then $e-y=2y$. Thus $e=3(\frac{w^2/(3e)+e}{4})$, so $w^2=e^2$, or $w=\pm e$, which we dealt with in Cases 2 and 3.

Condensing, we have that Equation (6) corresponds to a solution of the original problem as long as $w \notin \{0, e, -e, 3e, -3e\}$. The valid solutions generated classify all quartic polynomials with relatively prime integral coefficients and no constant term possessing three relative extrema and two points of inflection at lattice points whose point of inflection with greater abscissa is at the origin. To get back to the original problem, we add the parameters E for vertical displacement, r for horizontal displacement, and A for the greatest common divisor of A, B, C, and D. Substituting back through the equations now completes the proof of the following theorem.

THEOREM 1. *A quartic polynomial with integral coefficients has three relative extrema and two points of inflection at lattice points if and only if it is of the form*

$$A(x+r)^4 + A(\frac{(tv(3e))^2}{3e}+e)(x+r)^3 + A(-\frac{(tv(3e))^6}{4(3e)^3}+\frac{(tv(3e))^4}{6e}-\frac{3e(tv(3e))^2}{4})(x+r)+E,$$

w here A, E, e, r, $t \in \mathbb{Z}$, $tv(3e) \notin \{0, |e|, |3e|\}$, and $t>0$ is even if $4|e$, odd otherwise.

For example, if we take $A=1$, $E=0$, $r=0$, and $e=1$, then t can be any odd integer larger than 1 since $v(3e)=|3e|$. With $t=3$, we get $x^4+28x^3-3888x$, whose first and second derivatives have roots at

6, −9, −18, 0, and −14.

Conclusion While this problem has been completely solved, many related problems remain open. The most obvious generalization is to look for higher degree polynomials with the maximum possible number of relative extrema and points of inflection at lattice points. For quintic polynomials, we can no longer safely assume that the leading coefficient is 1; it can also be 3, as in $3x^5 - 250x^3 + 735x$. Even more general would be to demand that third derivatives and beyond must continue to have integral roots distinct from all the others. It is not known whether or not a nice quartic polynomial exists [3].

In the case of the quartic we can see that $f'''(x) = 24Ax + 6B = 24x + 24y$, so its root is $-y = -\frac{(tv(3e))^2/(3e)+e}{4}$. This root is always an integer when the other roots are, and turns out to be distinct. The classification of all nice cubic polynomials thus follows almost immediately as a corollary by taking the derivative with respect to x of the expression in the theorem and allowing A to be rational as long as all of the coefficients are integers.

COROLLARY 2. *A cubic polynomial is nice if and only if it is of the form*

$$4A(x+r)^3 + 3A(\frac{(tv(3e))^2}{3e} + e)(x+r)^2 + A(-\frac{(tv(3e))^6}{4(3e)^3} + \frac{(tv(3e))^4}{6e} - \frac{3e(tv(3e))^2}{4}),$$

where e, r, $t \in \mathbb{Z}$, $A \in \mathbb{Q}$ such that all the coefficients are integers, $tv(3e) \notin \{0, |e|, |3e|\}$, and $t > 0$ is even if $4|e$, odd otherwise.

References

[1] Tom Bruggeman and Tom Gush, Nice cubic polynomials for curve sketching, Mathematics Magazine 53 (1980), 233–234.

[2] Jim Buddenhagen, Charles Ford, and Mike May, Nice cubic polynomials, Pythagorean triples and the law of cosines, Mathematics Magazine 65 (1992), 244–249.

[3] Richard K. Guy, A quarter century of monthly unsolved problems, 1969–1993, *Amer. Math. Monthly* 100 (1993), 945–949.

[4] O.P. Lossers, Integer roots of cubics, *Amer. Math. Monthly* 96 (1989), 841–842.