Chain Lengths in the Dominance Lattice
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Abstract

We construct the largest union of two or three chains in the lattice of partitions of n under the dominance order. This construction provides a framework for finding the largest union of four or more chains. We also see how these chain lengths relate to a theorem of Gansner and Saks and the problem of finding the largest antichain.

1 Introduction

Let P_n denote the poset of partitions of the positive integer n, ordered by dominance (or majorization). That is, $\mu \leq \nu$ if $\mu_1 + \mu_2 + \cdots + \mu_k \leq \nu_1 + \nu_2 + \cdots + \nu_k$ for all k, where parts are indexed in nonincreasing order and trailing zeros are added as needed. This poset is a lattice and is self-dual under conjugation of partitions, where conjugation is accomplished by transposition of the Young diagram. The poset P_n is not graded (ranked) for $n \geq 7$, since there exist maximal chains from (n) to (1^n) of all lengths from $2n-3$ to $cn^{3/2}$ [2, 9].

Given any poset P, there exists a partition $\lambda(P)$ such that the sum of the first k parts of λ is the maximum number of elements in a union of k chains in P. In fact, the conjugate of λ has the same property for antichains [1, 5, 7, 8]. Let $\lambda_k(P)$ denote the kth part of this partition. The length $h(P_n)$ of the longest chain in P_n has been known for some time. If $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$, then $h(P_n) = \frac{m^3-m}{3} + rm$ [9]. In other words, $\lambda_1(P_n) = \frac{m^3-m}{3} + rm + 1$. Our main results are the following theorems.

Theorem 1. *For $n > 16$, $\lambda_2(P_n) = \lambda_1(P_n) - 6$.*

Theorem 2. *For $n > 22$, $\lambda_3(P_n) = \lambda_2(P_n) - 6$.*

We can define a rank function r on a graded poset such that if $x \leq y$, then $r(y) - r(x)$ is the length of a maximal chain from x to y. The Hasse diagram of a ranked poset can have all of the elements of the same rank on a horizontal line, giving every element a well-defined level. An unranked poset can also have this property, as illustrated in Figure 1.

In such a poset, we can draw a Hasse diagram where every element (except those on the top and bottom levels) covers something on the level below and is covered by something on the

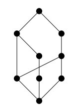

Figure 1: An unranked but leveled poset.

level above, thus giving each element a well-defined level on which to be drawn. Call such a poset *leveled*. Every element of a leveled poset P is in one of the longest chains of P.

Consider the subposet Q_n of P_n consisting of the partitions that appear in chains of length $h(P_n)$. Conjugation of partitions shows that Q_n is self-dual, since conjugation takes a decreasing chain to an increasing chain of the same length. Whether Q_n is a graded lattice remains an open question, but for our purposes it will suffice to use a weaker statement: for $\mu \in Q_n$, define $r(\mu)$ to be the length of the longest chain from (n) down to μ; for $\mu \notin \{(1^n), (n)\}$, μ is covered by an element ν such that $r(\nu) = r(\mu) - 1$ and covers an element ρ such that $r(\rho) = r(\mu) + 1$. In other words, Q_n is leveled. Note that the top element is level 0, and the levels increase as we move down.

Given any poset, the subposet of elements that appear in chains of maximum length will be leveled, and this is a maximal leveled subposet. It is maximal in the sense that it has the largest possible number of levels and the addition of any other elements would make it no longer leveled. This observation turns out to be useful for computing Greene-Kleitman partitions of certain posets [3, 4]. By temporarily discarding all elements that are not on longest chains, we can focus on the elements that really matter for our computations. We will know that the chains we find really are the longest possible, despite the lack of a true rank function, by showing that there is an antichain decomposition that puts the discarded elements onto levels already used in the chains.

A nonconstructive proof of Theorem 1 was given in [3] using an elegant lemma.

Lemma 1. *If a leveled poset has at least two elements on every level, then it has two disjoint chains of maximum length.*

The analogous lemma for three chains is less elegant to state, does not apply to all leveled posets, is tedious to prove, and does not generalize in any usable way to four or more chains. Moreover, the version of Theorem 2 stated in [3] is only for $n > 135$, so the proof given here makes a significant improvement on the lower bound, to the point that we can now compute $\lambda_3(P_n)$ for all n. The proofs of Theorems 1 and 2 we will give are constructive. While some casework is required to obtain the smallest n, we will see that an underlying pattern can be exhibited for large values of n without the cases, allowing some progress toward generalizing to unions of more than three chains.

The covering pairs in P_n come in two types. We say μ covers ν by an *H-step* if there exists i such that $\nu_i = \mu_i - 1$, $\nu_{i+1} = \mu_{i+1} + 1$, and $\nu_k = \mu_k$ for $k \notin \{i, i+1\}$. In terms of Young diagrams, this corresponds to moving a box one row down (and some distance to the left).

The other type is a *V-step*, which is an H-step on the conjugate, and corresponds to moving a box one column to the left (and some distance down). Chains from (n) to (1^n) consisting of H-steps followed by V-steps are maximal [9]. Not all maximal chains are of this form, however, so when we intersperse H and V steps it will be necessary to keep track of levels carefully.

2 Proof of Theorem 1

We will prove Theorem 1 by showing that there exist disjoint chains in Q_n of lengths $h(P_n)$ and $h(P_n) - 6$. Since Q_n is a subposet of P_n, these are also chains in P_n. For $n \geq 5$, the top three levels of P_n (and Q_n) consist only of (n), $(n-1, 1)$, and $(n-2, 2)$; the bottom three levels will be (1^n), $(2, 1^{n-2})$, and $(2, 2, 1^{n-4})$. Since these six elements of P_n are in maximal antichains of size 1 (that is, each of those six partitions is comparable to any other partition in P_n, so they cannot appear in antichains of 2 elements), the two chains will have the largest possible union, thus giving $\lambda_2(P_n)$.

To that end, we seek disjoint chains in Q_n from $(n-2, 1, 1)$ and $(n-3, 3)$ to $(2, 2, 2, 1^{n-6})$ and $(3, 1^{n-3})$. Let Q_n^* denote Q_n without the top three and bottom three elements, so our goal is to construct disjoint maximal chains in Q_n^*. Since Q_n^* is self-dual, we will construct chains to the halfway point that end with a self-conjugate partition or a partition that covers its conjugate when the number of elements in each chain is odd or even, respectively. The rest of the chain is then the conjugate of the chain already constructed.

Since $\lambda_1(P_n) = \frac{m^3-m}{3} + rm + 1$, and $\frac{m^3-m}{3}$ is even, and $\lambda_2(P_n)$ has the same parity as $\lambda_1(P_n)$ (since they differ by 6 in the cases we are considering), the number of levels in Q_n^* will be even only when m and r are both odd. If m or r is even, then any self-conjugate partitions in Q_n^* will be on its middle level (note that there can be non-self-conjugate partitions on the middle level whose conjugates are also on that level).

As a first approximation of the desired chains, take the following construction.

Chain L starts at $(n-2, 1, 1)$. At every step, we take the bottommost (in the Young diagram) possible H-step. That is, keep the large parts intact as long as possible, so the next few partitions are $(n-3, 2, 1)$, $(n-4, 3, 1)$, $(n-4, 2, 2)$, and $(n-4, 2, 1, 1)$.

Chain R starts at $(n-3, 3)$. At every step, we take the topmost possible H-step. That is, keep the number of parts as small as possible, so the next few partitions are $(n-4, 4)$, $(n-5, 5)$, $(n-6, 6)$, and $(n-7, 7)$.

Both chains will reach $(m, m-1, \ldots, r+1, r, r, r-1, \ldots, 2, 1)$, which is at least the halfway point, so only minor modifications will be needed to turn these into the desired chains. The following lemmas will help us ensure that our chains are disjoint.

Lemma 2. *If $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in R$, then $\mu_i - \mu_{i+1} \leq 2$ for $1 \leq i \leq k-2$. In other words, only the last difference can be greater than 2. Moreover, excluding the last difference,*

μ cannot have more than one difference equal to 2.

Proof. By construction, we are always using the topmost possible H-step. At first there is nothing to prove, since $k = 2$ through $\left(\frac{n}{2}, \frac{n}{2}\right)$ or $\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$. Think in terms of partition diagrams as in the definition of H-steps. If there are no differences greater than 1 (excluding the last one), then push one box from μ_{k-1} to increase the last part (or from μ_k to increase the number of parts). Now move to the left, pushing one box at a time until $\mu_i - \mu_{i+1} < 2$ for $i = 1, 2, \dots, k-2$ again. We never get a difference greater than 2 or more than one difference of 2 unless we had one before, so the result follows by induction. □

Lemma 3. *If $\mu = (\mu_1, \mu_2, \dots, \mu_k) \in L$, then $\mu_i - \mu_{i+1} \leq 2$ for $1 \leq i \leq k-2$. In other words, only the first difference can be greater than 2. Moreover, excluding the first difference, μ cannot have more than one difference equal to 2.*

The proof of Lemma 3 is similar to the proof of Lemma 2.

Recall that n can be expressed as $\binom{m+1}{2} + r$ where $0 \leq r \leq m$. The chains we use to prove Theorem 1 will depend on the value of r. We will see in the proofs of Theorems 1 and 2 that large values of r are relatively easy to work with, while the small values can be settled on a case-by-case basis. Fewer cases could be used if we were not trying to optimize the lower bound on n. The motivation behind these constructions was to find where the chains start to exist for each r, then find a description of the chains for the smallest m where they exist that generalizes for larger values of m.

If $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ is reachable from (n) by only H-steps, such as the elements of L and R, then $r(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$, since each box in μ_i had to be moved down $i-1$ times. Any $\mu \in L$ with $\mu_1 - \mu_2 > 2$ is not in R by Lemma 2. This construction yields the following lemma.

Lemma 4. *Chain L makes it safely to the partition $(m + r, m - 1, m - 2, \dots, 3, 2, 1)$ for $r \geq 2$, as shown in Figure 2.*

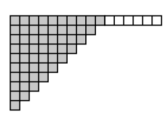

Figure 2: $m = 10$, $r = 6$

We conclude that the number of steps from (n) down to $(m + r, m - 1, m - 2, ..., 1)$ is $\sum_{i=1}^{m-1} i(m - i)$. The sum counts the triples from $\{1, ..., m + 1\}$, grouped by the middle element, and hence it equals $\binom{m+1}{3} = \frac{m^3-m}{6}$. That leaves $\left\lfloor \frac{rm}{2} \right\rfloor$ further steps down to the middle level.

Up to this point, the proof has been the same as in [3]. Instead of merely showing that L and R reach the middle level without intersecting, however, we now want to ensure that they will each form a chain with their conjugates in the lower half. When modifications are needed, call the revised chains L^* and R^*. We start with the construction of L^*.

We proceed to move half of the r excess boxes from the first row to the first column. If r is odd, then there will be one box to leave in the middle. More precisely, we use m H-steps to $(m+r-1, m-1, m-2, \ldots, 3, 2, 1, 1)$, another $m-1$ to $(m+r-2, m-1, m-2, \ldots, 3, 2, 2, 1)$, then a V-step to $(m+r-2, m-1, m-2, \ldots, 3, 2, 1, 1, 1)$. If r is even, then we continue alternating $m-1$ H-steps with a V-step until we reach $(m+r/2, m-1, m-2, \ldots, 3, 2, 1^{1+r/2})$, as shown in Figure 3.

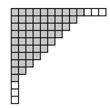

Figure 3: $m = 10$, $r = 6$

This is a self-conjugate partition on level $\frac{m^3-m}{6} + \frac{rm}{2}$, which is exactly what we need. Note that when $r \geq 4$, we safely avoid R by having a difference of at least 3 or two differences of 2 between the first parts of our partition at all times. Thus we have constructed L^* when r is even and $r \geq 4$.

When r is odd and at least 5, and m is even, we similarly arrive at $(m+1+(r-1)/2, m-1, m-2, \ldots, 3, 2, 1^{1+(r-1)/2})$ after $\frac{(r-1)m}{2} = \frac{rm}{2} - \frac{m}{2}$ steps. Now move one more box from the first part $\frac{m}{2}$ steps to arrive at the self-conjugate partition

$$\left(m + \frac{r-1}{2}, m-1, m-2, \ldots, \frac{m}{2}+2, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 3, 2, 1^{1+(r-1)/2}\right),$$

as shown in Figure 4.

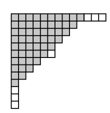

Figure 4: $m = 10$, $r = 7$

When $r \geq 5$ is odd and m is odd, we again take $\frac{(r-1)m}{2}$ steps to $(m+1+(r-1)/2, m-1, m-2, \ldots, 3, 2, 1^{1+(r-1)/2})$, but now we move the last box $\frac{m-1}{2}$ steps (for a total of $\lfloor \frac{rm}{2} \rfloor$, as needed) to arrive at

$$\left(m + \frac{r-1}{2}, m-1, m-2, \ldots, \frac{m+1}{2}+2, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, \ldots, 2, 1^{1+(r-1)/2}\right),$$

which covers its conjugate,

$$\left(m + \frac{r-1}{2}, m-1, m-2, \ldots, \frac{m+1}{2}+2, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \ldots, 2, 1^{1+(r-1)/2}\right),$$

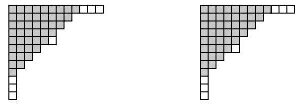

Figure 5: $m = 9$, $r = 7$

as shown in Figure 5.

We still need to construct L^* when $r = 0, 1, 2,$ or 3. First we construct R^*. Note that R goes through partitions of the form $(a, a-1, a-2, \ldots, a-k)$ where at most one part is repeated. This will be the last element of R with $k+1$ parts (or k parts if there is no repetition). Taking the partition of that form with $a = m+1$ and k as large as possible, as in Figure 6 for $n = 32$, we begin our construction of R^*.

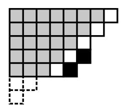

Figure 6: $m = 7$, $r = 4$

We will keep R^* disjoint from L^* for $r \geq 4$ by keeping the first difference equal to 1 (while the elements of L^* all had first difference at least $\lfloor \frac{r}{2} \rfloor$ at all times). We move the boxes shown in black in Figure 6, if any, to begin filling in the dotted area one row at a time. This will leave the gray and white boxes untouched. Now use white boxes starting from the bottom, as needed, to finish filling in the remaining dotted area one row at a time.

Thus R^* arrives at $(m+1, m, m-1, \ldots, m-r+2, m-r, m-r-1, \ldots, 2, 1)$, illustrated in Figure 7 for $n = 61$, at level $(1(m-1) + 2(m-2) + 3(m-3) + \cdots + (m-1)(1)) + (1 + 2 + \cdots + (r-1)) = \binom{m+1}{3} + \binom{r}{2} = \frac{m^3-m}{6} + \frac{r^2-r}{2}$.

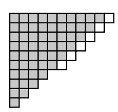

Figure 7: $m = 10$, $r = 6$

From here we construct R^* by moving half of the excess boxes (that is, the ones that are not part of $(m, m-1, m-2, \ldots, 2, 1)$) to the bottom one at a time. Note that R^* will not satisfy Lemma 2 since it will at times have two differences equal to 2, but since the first two differences will remain at 1 (at least for $r \geq 4$) we are still safe from intersecting L^*. If $r \geq 4$ is even, then we reach the self-conjugate partition (in Figure 8)

$$\left(m+1, m, m-1, \ldots, m - \frac{r}{2} + 2, m - \frac{r}{2}, m - \frac{r}{2} - 1, \ldots, \frac{r}{2}, \frac{r}{2}, \frac{r}{2} - 1, \ldots, 2, 1\right).$$

Since we used only H-steps, we do not need to compute the level. Similarly if $r \geq 5$ is odd and m is even, then we move $\frac{r-1}{2}$ boxes as before and one more box to the middle to reach the self-conjugate partition (Figure 8)

$$\left(m+1,\dots,m-\frac{r-1}{2}+2,m-\frac{r-1}{2},\dots,\frac{m}{2}+1,\frac{m}{2}+1,\frac{m}{2}-1,\dots,\frac{r-1}{2},\frac{r-1}{2},\dots,2,1\right).$$

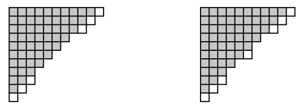

Figure 8: $m = 10$, $r = 6$ and $r = 7$

If $r \geq 5$ is odd and m is odd, then we similarly arrive at $\left(m+1,\dots,m-\frac{r-1}{2}+2,m-\frac{r-1}{2},\dots,\frac{m+1}{2}+1,\frac{m+1}{2}+1,\frac{m+1}{2}-1,\dots,\frac{r-1}{2},\frac{r-1}{2},\dots,2,1\right)$, which covers its conjugate, $\left(m+1,\dots,m-\frac{r-1}{2}+2,m-\frac{r-1}{2},\dots,\frac{m-1}{2}+1,\frac{m-1}{2}+1,\frac{m-1}{2}-1,\dots,\frac{r-1}{2},\frac{r-1}{2},\dots,2,1\right)$.

The cases where $r < 4$ remain. When $r = 3$, L reaches $(m+3, m-1, m-2, \dots, 3, 2, 1)$ by Lemma 4. The next three steps are $(m+2, m, m-2, m-3, m-4, \dots, 3, 2, 1)$, $(m+2, m-1, m-1, m-3, m-4, \dots, 3, 2, 1)$, and $(m+2, m-1, m-2, m-2, m-4, \dots, 3, 2, 1)$. The elements of L and R^* on this level are depicted in the top of Figure 9. Some care is needed now to ensure that we do not end up with both chains ending up at the middle level with first part $m+1$, ending in two 1s, and with the third excess box in the middle. Since L will go there naturally, we continue R^* by moving each of the three excess boxes in the next three steps to $(m, m, m-1, m-2, m-4, \dots, 2, 1)$. Now construct the rest of R^* by keeping the first two parts equal to m, which will keep us disjoint from L where the first part is at least $m+1$, moving the bottom excess box to make the last part 2, then moving the middle excess box to the middle. If m is even, then our self conjugate partitions will be $(m+1, m-1, \dots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \dots, 2, 1, 1)$ on L and $(m, m, m-2, \dots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \dots, 2, 2)$ on R^*, as shown in the bottom of Figure 9.

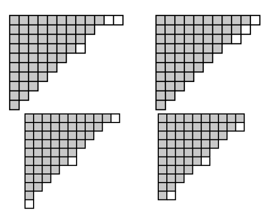

Figure 9: $m = 10$, $r = 3$

If m is odd, then as before we get $(m+1, m-1, \dots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, \dots, 2, 1, 1)$ on L and $(m, m, m-2, \dots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, \dots, 2, 2)$ on R^* covering their conjugates, namely $(m+1, m-1, \dots, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \dots, 2, 1, 1)$ on L and $(m, m, m-2, \dots, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \dots, 2, 2)$ on R^*. We used only H-steps, so our chains have the correct length. Note that we need $m \geq 4$ to make this construction work, which is fine since $m = 4$, $r = 3$ gives $n = 13$.

When $r=2$, L reaches $(m+2, m-1, m-2, \ldots, 3, 2, 1)$ by Lemma 4. At the same level, R^* reaches $(m+1, m, m-2, m-3, \ldots, 4, 3, 3)$, as shown in the top row of Figure 10. We have m steps to go. To keep the chains apart, we use L and continue R^* by moving each of the two excess (white) boxes in the next two steps to $(m, m, m-1, m-3, m-4, \ldots, 4, 3, 3)$, giving the bottom row of Figure 10.

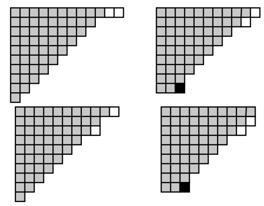

Figure 10: $m=10$, $r=2$

Now we can safely move the black box into position, so that the next element of R^* will be $(m, m, m-1, m-3, m-4, \ldots, 4, 3, 2, 1)$, then move the white boxes down (move the bottom one a step, then the top one, then repeat as needed) until we reach a self-conjugate partition. L will go to $(m+1, m-1, m-2, \ldots, 2, 1, 1)$ while R^* reaches $(m, m-1, \ldots, \frac{m}{2}+2, \frac{m}{2}+2, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}-2, \ldots, 2, 1)$ if m is even or $(m, m-1, \ldots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \ldots, 2, 1)$ if m is odd, as shown in Figure 11. Here we need $m > 4$, so the largest case that does not work is $m=4$ and $r=2$, or $n=12$.

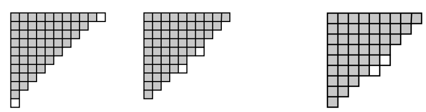

Figure 11: $m=10$, $r=2$ and R^* for $m=9$, $r=2$

When $r=1$, both L and R^* go to $(m+1, m-1, m-2, \ldots, 2, 1)$, so we back up one step to $(m+2, m-2, m-3, \ldots, 2, 1)$ on L and $(m+1, m-1, m-2, \ldots, 4, 3, 3)$ on R^*, as shown in the left half of Figure 12. From here, L continues without modification, while in R^* we move the white excess box to the middle. When m is even, after $\frac{m}{2}$ steps we are one step from the middle level with $(m, m-1, \ldots, \frac{m}{2}+2, \frac{m}{2}+2, \frac{m}{2}, \frac{m}{2}-1, \ldots, 2, 1)$ on L and $(m, m-1, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 4, 3, 3)$ on R^*. One more step, an H-step for L and a V-step for R^*, gets us to the self conjugate partitions $(m, m-1, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 2, 1)$ on L and $(m-1, m-1, m-1, m-3, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 4, 3, 3)$ on R^*, as shown in the right half of Figure 12.

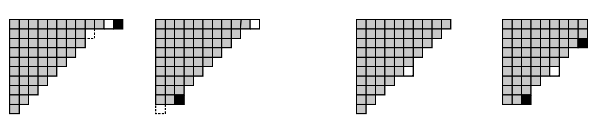

Figure 12: $m=10$, $r=1$

If m is odd, then we similarly get to $(m, m-1, \ldots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, \ldots, 2, 1)$ on L and $(m-1, m-1, m-1, m-3, \ldots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, \ldots, 4, 3, 3)$ on R^*

covering their conjugates, namely $(m, m-1, \ldots, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \ldots, 2, 1)$ on L and $(m-1, m-1, m-1, m-3, \ldots, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, \ldots, 4, 3, 3)$ on R^*.

We need $m > 5$ for this construction to work. When $m = 5$ and $r = 1$ we have $n = 16$, which is the largest n for which $\lambda_2(P_n) \neq \lambda_1(P_n) - 6$. When $m = 6$, the self-conjugate partition on R^* is $(5, 5, 5, 4, 3)$ since the middle box increases one of our 3s to a 4. A similar overlap occurs when $m = 7$, where we find $(6, 6, 6, 5, 3, 3)$ covering its conjugate $(6, 6, 6, 4, 4, 3)$.

When $r = 0$, since L and R first meet at $(m, m-1, m-2, \ldots, 3, 2, 1)$ on the middle level, we back up a step to $(m+1, m-2, m-2, \ldots, 3, 2, 1)$ on L and $(m, m-1, \ldots, 4, 3, 3)$ on R. Now continue with V-steps to the self-conjugate partitions $(m+1, m-2, m-2, \ldots, 3, 1, 1, 1)$ on L^* and $(m-1, m-1, m-1, m-3, \ldots, 4, 3, 3)$ on R^*. See Figure 13.

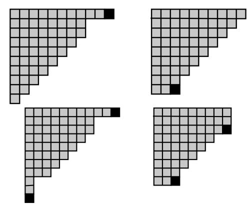

Figure 13: $m = 10$, $r = 0$

We need $m \geq 5$ for both partitions in Figure 13 to exist, which first happens at $n = 15$, thus completing the proof. By letting R^* go to $(4, 3, 2, 1)$ we can also construct the two chains when $n = 10$ $(m = 4)$. See Table 1 for more data about the small cases.

3 Proof of Theorem 2

Once again we wish to construct disjoint chains to the middle level. We use the chains L^* and R^* constructed in the proof of Theorem 1 (which were carefully constructed with Theorem 2 in mind), plus a new chain, called M (for middle), which will start at $(n-5, 4, 1)$.

We will try to keep M disjoint by keeping the second difference greater than 2, or having two differences equal to 2 somewhere in the middle, so as to avoid the original L and R by Lemmas 2 and 3. To take the first few steps along the middle chain without intersecting the right chain, we need $n > 14$ so that we can go through $(n-5, 4, 1)$, $(n-6, 5, 1)$, $(n-7, 6, 1)$, $(n-7, 5, 2)$, and $(n-7, 5, 1, 1)$, after which we keep the second difference greater than 2 as long as possible.

Take the bottommost possible H-step that keeps the second difference greater than 2 (or, if that is not possible, the second and third differences equal to 2), thus avoiding L and R by Lemmas 2 and 3. Thus M arrives safely at $(m+r-2, m+1, m-2, m-3, \ldots, 2, 1)$ at level $\frac{m^3-m}{6} + 2$ for $r \geq 4$. Figure 14 shows this partition for $n = 63$.

For large values of r, we can leave the two excess boxes in the second row alone and move

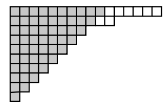

Figure 14: $m = 10$, $r = 8$

$\frac{r}{2}$ boxes from the first row down to form the self-conjugate partition $(m + \frac{r-4}{2}, m+1, m-2, m-3, \ldots, 3, 2, 2, 2, 1^{(r-6)/2})$ if r is even or $(m + \frac{r-5}{2}, m+1, m-2, m-3, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, 3, 2, 2, 2, 1^{(r-7)/2})$ if r is odd and m is even; see Figure 15.

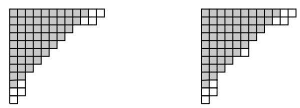

Figure 15: $m = 10$, $r = 8$ and $r = 9$

If m and r are both odd, then we get to
$\left(m + \frac{r-5}{2}, m+1, m-2, m-3, \ldots, \frac{m+1}{2}+1, \frac{m+1}{2}+1, \frac{m+1}{2}-1, 3, 2, 2, 2, 1^{(r-7)/2}\right)$,
which covers its conjugate,
$\left(m + \frac{r-5}{2}, m+1, m-2, m-3, \ldots, \frac{m-1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}-1, 3, 2, 2, 2, 1^{(r-7)/2}\right)$.

These constructions work for $r \geq 8$ and avoid both L^* and R^*. Since we used some V-steps, we can verify the level by noting that each white box in the first column took m steps to get there, and each one in the second column took $m-1$ steps. When m and r are even, the total number of steps past level $\frac{m^3-m}{6}$ is $2 + \left(\frac{r-4}{2}\right)m + 2(m-1) = \frac{rm}{2}$, as desired; the level calculation works similarly for the other parities of m and r.

For $r = 6$, we only make it as far as $(m+2, m+1, m-2, \ldots, 3, 2, 2, 1)$. We then need to move the second excess box in the second row down to make room to move the second excess box down from the first row with an H-step, which gets us to the self-conjugate partition $(m+1, m+1, m-2, \ldots, 3, 2, 2, 2)$. See Figure 16.

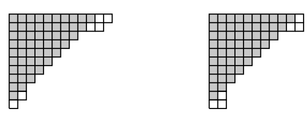

Figure 16: $m = 10$, $r = 6$

The case $r = 7$ is similar, with another box in the middle, but with the usual cases depending on the parity of m.

The cases where $r < 6$ require ad hoc constructions to minimize m.

For $r = 4$, start at $(m+2, m+1, m-2, \ldots, 2, 1)$. We cannot move two boxes to the bottom to make either of the "obvious" self-conjugate partitions since they were already used in L^* and R^*. Instead we move all four excess boxes to the middle (one at a time) to get the self-conjugate partition

$$\left(m, m-1, \ldots, \frac{m}{2}+3, \frac{m}{2}+3, \frac{m}{2}+2, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}-1, \frac{m}{2}-3, \ldots, 2, 1\right)$$

if m is even and

$$\left(m, m-1, \ldots, \frac{m+1}{2}+2, \frac{m+1}{2}+2, \frac{m+1}{2}+1, \frac{m-1}{2}+1, \frac{m-1}{2}, \frac{m-1}{2}-2, \ldots, 2, 1\right)$$

if m is odd. See Figure 17. Note that we need $m \geq 5$ to keep M disjoint from L^* and R^*.

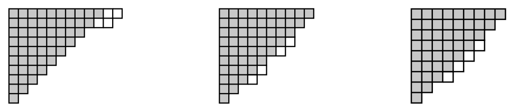

Figure 17: $r = 4$

The case where $r = 5$ works essentially the same as $r = 4$.

When $r = 3$, M makes it safely to $(m+1+(m-2), m+1, m-3, m-4, \ldots, 2, 1)$, as shown in the first part of Figure 18. The next of the dark gray boxes would normally fill in the bottom dotted box, but instead we must stop it at the previous row and continue filling in from there until we reach $(m+2, m+1, m-2, m-3, \ldots, 3, 2)$ (second part of Figure 18). Now we move a white box down to $(m+2, m, m-2, m-3, \ldots, 4, 3, 3)$ (third part of Figure 18).

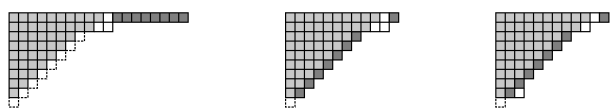

Figure 18: $m = 10$, $r = 3$

During this move, having the first difference and another one equal to 2 keeps M disjoint from R^* while ending in 3 and 2 (or 4 and 2) keeps it disjoint from L. Now ending in two 3s will keep M disjoint the rest of the way. Now move the three excess boxes from the top to the middle (the exact order is not crucial), then finally move the bottom white box down to fill in the dotted box. If m is even, then this will give us the self-conjugate partition $(m, m-1, \ldots, \frac{m}{2}+2, \frac{m}{2}+2, \frac{m}{2}+1, \frac{m}{2}, \frac{m}{2}-2, \ldots, 2, 1)$ shown in Figure 19. If m is odd, then we again get a partition that covers its conjugate. Note this construction requires $m > 5$, or $n > 18$.

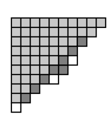

Figure 19: $m = 10$, $r = 3$

When $r = 2$, similar to the previous case, M makes it safely to $(m+(m-2), m+1, m-3, m-4, \ldots, 2, 1)$, as shown in the first part of Figure 20. We fill the bottom dotted box with one of the dark gray boxes, but the next one will stop short by a row at $(m+(m-4), m+1, m-3, m-4, \ldots, 4, 3, 1, 1)$, then keep filling in until reaching $(m+2, m+1, m-2, m-2, \ldots, 3, 1, 1)$ (second part of Figure 20). Now we move the outer white box to fill in the upper dotted box, then move the other white box down until reaching $(m+2, m-1, \ldots, 5, 4, 4, 1, 1)$ (third part of Figure 20). Next move one of the remaining dark gray boxes down until it is in the row above the white box, then move the other dark gray box down one row to give us $(m, m, m-2, \ldots, 5, 5, 4, 1, 1)$ (fourth part of Figure 20). Now we can move the white box down and the lower dark gray box down three times to reach the self-conjugate partition $(m, m, m-2, m-3 \ldots, 4, 3, 2, 2)$ (fifth part of Figure 20).

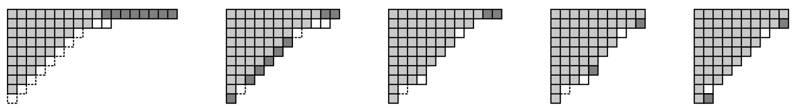

Figure 20: $m = 10$, $r = 2$

If the construction of R^* in this case had been more like the previous cases, with the excess boxes at the ends instead of the middle, then we leave it as an exercise for the reader to figure out why it would not have been possible to construct a third chain for $n = 23$.

When $r = 1$, similar to the second part of Figure 20 in the previous case, M can get to $(m+2, m+1, m-3, m-4, m-4, m-5, \ldots, 4, 3, 1, 1)$ (first part of Figure 21). Since neither L nor R^* contains partitions ending in two 1s near this level, we can safely fill in the upper two dotted boxes with the excess boxes in the second row, arriving at $(m+2, m-1, \ldots, 3, 1, 1)$ (second part of Figure 21). For m even, we now move the black box down and the dark gray box halfway down to get to $(m, m-1, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 4, 4, 1, 1)$ at level $\frac{m^3-m}{6}-1+\frac{m}{2}$ (third part of Figure 21). Since we are one step away from the middle, we can take a V-step to the self-conjugate partition $(m, m-2, m-2, m-2, m-4, \ldots, \frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}-1, \ldots, 4, 4, 1, 1)$ (fourth part of Figure 21).

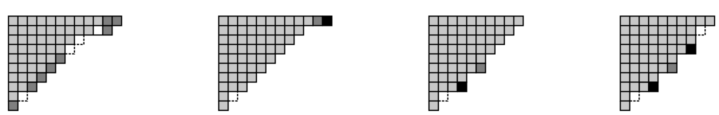

Figure 21: $m = 10$, $r = 1$

If m is odd, then we similarly construct a partition that covers its conjugate. This construction works for $m > 7$, or $n > 29$. A separate construction is needed for $n = 29$ since there is no third chain with the upper half conjugate to the lower half. We leave that case as an exercise. The largest case that does not work at all is $m = 6$, or $n = 22$.

When $r = 0$, similar to the first part of Figure 21 in the previous case, M can get to $(m+2, m+1, m-3, m-4, m-5, m-5, \ldots, 4, 3, 1, 1)$ (first part of Figure 22). The ending of two 1s at this level again keeps M disjoint from the previous two chains, so we can safely fill in the upper three dotted boxes to reach $(m+1, m-1, \ldots, 3, 1, 1)$ (second part of Figure 22), then move the final excess box down to get the self-conjugate partition $(m, m-1, \ldots, 2, 1)$ (third part of Figure 22).

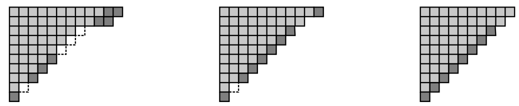

Figure 22: $m = 10$, $r = 0$

This construction (or a minor variation on it for small m) gives us a third chain for $m \geq 5$, or $n \geq 15$. Across all cases, the largest value of n for which we could not find a third chain came from $m = 6$ and $r = 1$, or $n = 22$, thus completing the proof of Theorem 2.

4 Smaller cases and related questions

The smaller n for which $\lambda_2(P_n) = \lambda_1(P_n) - 6$ are 10, 13, 14, and 15. More generally, Table 1 shows the Greene-Kleitman partitions for P_n when $1 \leq n \leq 14$, computed by explicit chain and antichain decomposition. In all of these cases, the elements added between $\lambda_{k-1}(P_n)$ and $\lambda_k(P_n)$ form a chain that is added to the previous $k-1$ chains (and similarly for antichains). The proofs of Theorems 1 and 2 show that this is the case for all n when $k \in \{2, 3\}$; it would be interesting to know if it holds for all k.

n	$\lambda(P_n)$
1	(1)
2	(2)
3	(3)
4	(5)
5	(7)
6	$(9, 2)$
7	$(12, 3)$
8	$(15, 7)$
9	$(18, 9, 3)$
10	$(21, 15, 4, 2)$
11	$(25, 18, 10, 3)$
12	$(29, 21, 13, 10, 4)$
13	$(33, 27, 18, 14, 6, 3)$
14	$(37, 31, 24, 19, 15, 6, 3)$

Table 1: Known values of $\lambda(P_n)$.

We know from Dilworth's theorem that P_n can be covered by $\lambda_1(P_n)$ antichains. What we have shown is that we can do this with six one-element antichains, six two-element antichains, and the remaining antichains of size three or larger. Moreover, there does not exist such a covering with seven one-element antichains, nor with thirteen antichains of at most two elements (for n sufficiently large).

While $\lambda_k(P_n) - \lambda_{k+1}(P_n)$ need not always be 6, it appears that it should be constant for n sufficiently large.

Conjecture 1. *When n is sufficiently large, $\lambda_k(P_n) - \lambda_{k+1}(P_n)$ depends only on k.*

Theorems 1 and 2 verify Conjecture 1 for $k = 1$ and 2, respectively. As further evidence for Conjecture 1, we can use the general idea from the proofs of those theorems to sketch a proof of the following proposition.

Proposition 1. *When n is sufficiently large, $\lambda_k(Q_n) - \lambda_{k+1}(Q_n)$ depends only on k.*

Proof. Construct k disjoint chains by keeping the ith difference greater than 2 for $1 \leq i \leq k$. The first chain can start at (n), the second at $(n-3, 3)$, etc., with each chain starting wherever Q_n first becomes wide enough and the first few levels being whatever they need to be to keep the chains disjoint until the partitions all have more than k parts. We then use moves similar to the constructions of L^* and M (but not R) to guide the kth chain to $(m+m+r-3k+5, m+1, m, m-1, \ldots, m-k+4, m-k, m-k-1, m-k-2, \ldots, 2, 1)$ and then on to the self-conjugate partition $(m+r/2, m-1, m-2, \ldots, m-k+3, m-k+1, m-k+1, m-k+1, m-k-1, m-k-2, \ldots, k+1, k+1, k-2, k-2, k-3, \ldots, 2, 1^{1+r/2})$. See Figure 23, with an extra box in the middle if r is odd. If m is also odd, then instead of a self-conjugate partition we get a partition that covers its conjugate by moving the extra box in the middle. The rest of the chain is then the conjugate of the first half thus constructed. □

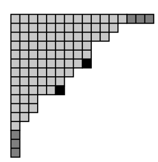

Figure 23: $m = 13$, $r = 6$, $k = 5$

The difficulty in going from Proposition 1 to Conjecture 1 lies in incorporating the unranked elements of P_n, which may appear in long chains starting at $k = 4$. It appears that $\lambda_4(P_n) = \lambda_3(P_n)$ while $\lambda_4(Q_n) = \lambda_3(Q_n) - 2$. Incorporating the unranked elements has been the bulk of the work done to prove analogous theorems for the Tamari lattice [4, 10].

Let M be the transition matrix from the bases $\{e_\mu\}$ to $\{m_{\mu'}\}$ of homogeneous symmetric functions of degree n. Since $M_{\mu\nu} > 0$ if and only if $\nu \leq \mu'$, it is a theorem of Gansner and Saks (independently) that a generic matrix with the same 0 entries will have Jordan blocks whose sizes are exactly the parts of $\lambda(P_n)$ [1, 6, 11]. Using Table 1 and Maple, one can verify that M has these Jordan block sizes $n \leq 14$. However, for $n = 15$ the partition from the

Jordan blocks is $(41, 35, 27, 25, 18, 14, 7, 5, 4)$, while $\lambda_3(P_{15}) = 29$ and $\lambda_9(P_{15}) = 2$. Thus the theorem of Gansner and Saks does not provide an alternate method of computing $\lambda(P_n)$ in general.

Another open problem is to find the size $a(n)$ of the largest antichain in P_n, which is the number of parts of $\lambda(P_n)$. Let $p(n)$ be the number of partitions of n. There is the obvious upper bound $a(n) \le p(n)$. By Dilworth's theorem, $a(n) \ge p(n)/(h(P_n)+1)$, so we have $\Omega(n^{-5/2}e^{\pi\sqrt{2n/3}}) \le a(n) \le O(n^{-1}e^{\pi\sqrt{2n/3}})$. It would be interesting to find a constructive proof that $a(n)$ is at least as large as the lower bound. In addition to the values of $a(n)$ implied by Table 1, we can find that $a(15) = 9$. Antichains of that length are $\{(7, 1^8)$, $(6, 2, 2, 1^5)$, $(5, 4, 1^6)$, $(5, 3, 2, 2, 1^3)$, $(5, 2^5)$, $(4, 4, 3, 1^4)$, $(4, 4, 2, 2, 2, 1)$, $(4, 3, 3, 3, 1, 1)$, $(3^5)\}$ and the set of their conjugates. One can also verify that $a(16) = 10$. The sequence $\{a(n) : n \ge 0\}$ is number A076269 and the sequence $\{\lambda_{a(n)}(P_n) : n \ge 0\}$ (the smallest part of $\lambda(P_n)$) is number A076779 in [12].

One construction that shows $a(n)$ has a lower bound of the form $e^{c\sqrt{n}}$ is as follows. Begin with the antichain $\{(7, 3, 2, 1^4)$, $(7, 2, 2, 2, 2, 1)$, $(6, 5, 1^5)$, $(6, 4, 2, 2, 1, 1)$, $(6, 3, 3, 2, 2)$, $(5, 5, 3, 1, 1, 1)$, $(5, 5, 2, 2, 2)$, $(5, 4, 4, 2, 1)$, $(4, 4, 4, 4)\}$ in P_{16}. Let $\nu + 7n$ denote a partition ν from the list with $7n$ added to each part. Consider ν to have 7 parts, so some of them might be 0. Now $(\nu_{(n)} + 7n, \nu_{(n-1)} + 7(n-1), \ldots, \nu_{(1)} + 7, \nu_{(0)})$, where each $\nu_{(i)}$ denotes one of the nine partitions in the list, is a partition of $N = 16(n+1) + 49\frac{n^2+n}{2} = \frac{49}{2}n^2 + O(n)$. The partitions thus generated are pairwise incomparable, hence we have an antichain of size 9^{n+1} in P_N. This yields a lower bound for $a(n)$ of $e^{c\sqrt{n}}$, where $c = \ln 9\sqrt{2/49} = 0.4439\ldots$. By starting with a 28-element antichain in P_{27}, where each ν has at most 9 parts and largest part at most 8, one can similarly get $c = \frac{\ln 28}{6} = 0.555\ldots$. This is still a long way from $\pi\sqrt{2/3} = 2.565\ldots$, but at least it is constructive.

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