Character Polynomials and Row Sums of the Symmetric Group

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Abstract

Character polynomials provide a combinatorial description of the characters of the irreducible representations of the symmetric group. Using generating functions and some facts about integer partitions, we show that certain row sums in the character table for the symmetric group are positive. The nonnegativity of these row sums is known from representation theory, but finding a combinatorial proof of positivity for all rows remains an open problem.

1 Introduction

The problem of finding a combinatorial interpretation of the row sums of the character table of the symmetric group S_n was posed by Richard Stanley as problem 12 in [6]. These sums are nonnegative since they give the multiplicity of an irreducible in the character of the action of S_n on itself by conjugation [2], but the fact that they are positive integers (except for one for S_2 whose sum is 0) inspires the search for a combinatorial explanation.

Let us first compute the character table for S_3 to put some perspective on what is being computed. An *n-dimensional representation* of a group is a homomorphism into $GL_n(\mathbb{R})$. In other words, a map f from the group to a set of matrices with real number entries such that $f(ab) = f(a)f(b)$ for all a and b in the group. The character of an element of the group is the trace, or sum of the diagonal entries, of its corresponding matrix. Trace is invariant under conjugation (replacing a group element a by bab^{-1} for some group element b), so the character table comprises the value of the character of each representation on each conjugacy class.

Conjugacy classes in S_n are determined by cycle types, which give partitions of n. Given any two representations of dimensions m and n, we can build an $(m+n)$-dimensional representation by simply putting the two matrices along the diagonal of an $(m+n) \times (m+n)$ matrix and filling in the rest with 0's. Representations that cannot be built out of other representations in this way are called *irreducible*. The number of irreducible representations of a group equals the number of conjugacy classes, thus the representations of S_n can also be indexed by partitions of n. The bijection between partitions and irreducible representations is not so simple, so let us instead exhibit three distinct representations for S_3. The first is the trivial representation, where every permutation maps to 1. In general, the trivial representation corresponds to the partition (n), and the sign representation, where even permutations map to 1 and odd permutations map to -1, corresponds to the partition of all 1's, denoted (1^n).

There is also a two-dimensional representation of S_3 that can be found via matrices of the linear transformations that permute the points $A = (1, 0)$, $B = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $C = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The identity permutation ABC has cycle type $(1, 1, 1)$ and maps to the identity matrix, with trace 2. The transposition ACB has cycle type $(2, 1)$ and maps to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflection across the y-axis), with trace 0. The cycle CAB has cycle type (3) and maps to $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ (120° counterclockwise rotation) with trace -1.

Indexing the rows by representation and the columns by conjugacy class, we can now write down the complete character table for S_3.

Character Table for S_3

	Identity (1,1,1)	Transposition (2,1)	Cycle (3)
Trivial (3)	1	1	1
2-D (2,1)	2	0	-1
Sign (1,1,1)	1	-1	1

For a more thorough introduction to the algebra concepts used here, see [1]. The character table for S_4 is also given here.

Character Table for S_4

	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
(4)	1	1	1	1	1
(3,1)	3	1	-1	0	-1
(2,2)	2	0	2	-1	0
(2,1,1)	3	-1	-1	0	1
(1,1,1,1)	1	-1	1	1	-1

2 Character Polynomials

For the row of the character table indexed by a partition whose largest part is $n - k$, the entries will depend only on the cycles of size at most k. Let a_k denote the number of cycles of length k in the cycle type. Then row (n) is all 1's, row $(n - 1, 1)$ is $a_1 - 1$, and rows $(n - 2, 2)$ and $(n - 2, 1, 1)$ have formulas in terms of a_1 and a_2.

Let λ be a partition of k. The character polynomial for the $(n-k,\lambda)$ row of the table is denoted by X^λ and is given by the following formula [4].

$$X^\lambda = \sum_{\rho,\sigma} (-1)^{l(\sigma)} z_\sigma^{-1} \chi^\lambda_{\rho\cup\sigma} \binom{a}{\rho}$$

The summation is over all ordered pairs of partitions ρ and σ such that $|\rho|+|\sigma|=|\lambda|$, $l(\sigma)$ denotes the length, or number of parts, of σ, $z_\sigma = \prod_{i\geq 1} i^{m_i} m_i!$ where $\sigma = (1^{m_1}, 2^{m_2}, ...)$, $\chi^\lambda_{\rho\cup\sigma}$ is the character from row λ, column $\rho\cup\sigma$ of $S_{|\lambda|}$, and $\binom{a}{\rho} = \prod_{r\geq 1} \binom{a_r}{n_r}$ where $\rho = (1^{n_1}, 2^{n_2}, ...)$.

For example, consider $(n-2,2)$ row of the character table, so $\lambda = (2)$. The following table lists all possible pairs of ρ and σ with their affiliated values.

ρ	σ	$(-1)^{l(\sigma)}$	z_σ^{-1}	$\chi^\lambda_{\rho\cup\sigma}$	$\binom{a}{\rho}$	summand
(2)	$\emptyset$	1	1	1	a_2	a_2
(1)	(1)	-1	1	1	a_1	$-a_1$
$\emptyset$	(2)	-1	$\frac{1}{2}$	1	1	$-\frac{1}{2}$
(1,1)	$\emptyset$	1	1	1	$\frac{a_1(a_1-1)}{2}$	$\frac{a_1(a_1-1)}{2}$
$\emptyset$	(1,1)	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$

Therefore $\chi^2 = a_2 - a_1 - \frac{1}{2} + \frac{a_1(a_1-1)}{2} + \frac{1}{2} = a_2 - a_1 + \frac{a_1(a_1-1)}{2}$. It is better for our purposes not to combine the two a_1 terms.

The same method can be used to derive character polynomials for any row of the character table. The table below shows several examples.

Partition	Polynomial
(n)	1
$(n-1,1)$	$a_1 - 1$
$(n-2,2)$	$a_2 - a_1 + \frac{a_1(a_1-1)}{2}$
$(n-2,1^2)$	$-a_2 - a_1 + 1 + \frac{a_1(a_1-1)}{2}$
$(n-3,3)$	$a_3 + a_1a_2 - a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6}$
$(n-3,2,1)$	$-a_3 - a_1(a_1-1) + \frac{a_1(a_1-1)(a_1-2)}{3} + a_1$
$(n-3,1^3)$	$a_3 - a_1a_2 + a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6} + a_1 - 1$
$(n-4,1^4)$	$-a_4 + a_1a_3 + \frac{a_2(a_2-1)}{2} - a_1\frac{a_2(a_2-1)}{2} - a_3 + a_1a_2 + \frac{a_1(a_1-1)(a_1-2)(a_1-3)}{24} - \frac{a_1(a_1-1)(a_1-2)}{6} - a_2 + \frac{a_1(a_1-1)}{2} + 1$

3 Row Sums

To calculate a row sum, we use generating functions to relate character polynomials to sums of the partition function $p(n)$. In particular, $\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$, where the power of x^i in the expansion of $\frac{1}{1-x^i}$ as a geometric series corresponds to the number if i's in a partition.

The row of the partition $(n-1, 1)$ has a character polynomial of $a_1 - 1$, i.e., each character in this row is equal to one less than the number of 1's in the partition of n from the cycle type. Because this character polynomial only depends on the number of 1's, we are only concerned with the $\frac{1}{1-x}$ part of the product. To extract this part, we introduce a variable u to expression:

$$\frac{1}{1-ux} = 1 + ux + u^2x^2 + u^3x^3 + u^4x^4 + \cdots$$

The next step is to take the partial derivative with respect to u and then set $u = 1$.

$$\frac{\partial}{\partial u}\frac{1}{1-ux}\bigg|_{u=1} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

The result is that now the coefficient of x^n counts the number of 1's, which implies that

$$\frac{\partial}{\partial u}\frac{1}{1-ux}\prod_{i=2}^{\infty}\frac{1}{1-x^i}\Bigg|_{u=1} = \sum_{n=0}^{\infty}\left(\sum_{\lambda\vdash n} a_1(\lambda)\right)x^n$$

is the generating function for the total number of 1's in all partitions of n. A similar argument proves the following lemma.

Lemma 1.

$$\frac{1}{j!}\frac{\partial^j}{\partial u^j}\frac{1}{1-ux^k}\prod_{i\neq k}\frac{1}{1-x^i}\Bigg|_{u=1} = \sum_{n=0}^{\infty}\left(\sum_{\lambda\vdash n}\binom{a_k(\lambda)}{j}\right)x^n.$$

We can also compute

$$\frac{\partial}{\partial u}\frac{1}{1-ux}\prod_{i=2}^{\infty}\frac{1}{1-x^i}\Bigg|_{u=1} = \frac{x}{(1-x)^2}\prod_{i=2}^{\infty}\frac{1}{1-x^i} = \frac{x}{1-x}\sum_{n=0}^{\infty}p(n)x^n$$

$$= \left(x + x^2 + x^3 + x^4 + \ldots\right)\sum_{n=0}^{\infty}p(n)x^n,$$

hence $\sum_{\lambda\vdash n} a_1(\lambda) = p(n-1)+p(n-2)+p(n-3)+\cdots$. Finally, the row sum for the $(n-1,1)$ row is $\sum_{\lambda\vdash n}(a_1(\lambda)-1) = -p(n)+p(n-1)+p(n-2)+p(n-3)+\cdots$. The positivity of this sum is established by the following lemma.

Lemma 2. *For $n \geq 2$, $p(n-1) \leq p(n) \leq p(n-1)+p(n-2)$, with strict inequality for $n \geq 5$.*

Proof. If a partition of n contains a 1, then removing a 1 provides a unique partition of $n-1$. If the partition does not contain any 1's but has at least two parts, then subtracting 1 from each of the two smallest parts yields a unique partition of $n-2$ with at least two parts, and the partition (n) can map to $(n-2)$. This provides a one-to-one map from partition of n to partitions of $n-1$ or $n-2$, and it is not onto for $n-2 \geq 3$ since we will not get any partitions of $n-2$ with at least three 1's.

For the first inequality, every partition of $n-1$ yields a distinct partition of n by appending a 1, but there will always be at least one more partition of n, namely (n), for $n > 1$. $\square$

By observing that $-p(4)+p(3)+p(2)+p(1)+p(0) = -5+3+2+1+1 = 2$, $-p(3)+p(2)+p(1)+p(0) = -3+2+1+1 = 1$, and $-p(2)+p(1)+p(0) = -2+1+1 = 0$, we thus establish the positivity of the $(n-1,1)$ row for all $n > 2$.

For the $(n-2,2)$ row the characters are $a_2 - a_1 + \frac{a_1(a_1-1)}{2}$, so by Lemma 1 the generating function for the row sum is

$$\frac{\partial}{\partial u}\frac{1}{1-ux^2}\prod_{i\neq 2}\frac{1}{1-x^i} - \frac{\partial}{\partial u}\frac{1}{1-ux}\prod_{i\neq 1}\frac{1}{1-x^i} + \frac{1}{2}\frac{\partial^2}{\partial u^2}\frac{1}{1-ux}\prod_{i\neq 1}\frac{1}{1-x^i}\Bigg|_{u=1}$$

$$= \left(\frac{x^2}{1-x^2} - \frac{x}{1-x} + \frac{x^2}{(1-x)^2}\right)\prod_{i=1}^{\infty}\frac{1}{1-x^i} = \left(\frac{x^2}{1-x^2} - \frac{x}{1-x} + \frac{x^2}{(1-x)^2}\right)\sum_{n=0}^{\infty}p(n)x^n.$$

Using the expansions $\frac{x^2}{1-x^2} = x^2+x^4+x^6+x^8+\cdots$, $\frac{x}{1-x} = x+x^2+x^3+x^4+\cdots$, and $\frac{x^2}{(1-x)^2} = x^2+2x^3+3x^4+4x^5+\cdots$, the row sum is therefore

$$\begin{gathered}(p(n-2)+p(n-4)+p(n-6)+p(n-8)+\cdots)\\ -(p(n-1)+p(n-2)+p(n-3)+p(n-4)+\cdots)\\ +(p(n-2)+2p(n-3)+3p(n-4)+4p(n-5)+\cdots)\\ = -p(n-1)+p(n-2)+p(n-3)+3p(n-4)+3p(n-5)+\cdots\end{gathered}$$

where the remaining terms are all positive. The first three terms show that this sum is positive for $n \geq 5$ by Lemma 2 and the smaller values of n are easily checked to be positive as well.

The computation appears to get trickier at $(n-3,3)$ due to the a_1a_2 term in the character polynomial, but this is handled by computing $\left.\frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{1}{1-ux}\frac{1}{1-vx^2}\prod_{i\geq 3}\frac{1}{1-x^i}\right|_{u=v=1}$ in analogy with Lemma 1. We thus get the following table of row sums.

Row	Row Sum
(n)	$p(n)$
$(n-1,1)$	$p(n-1)+p(n-2)+p(n-3)+p(n-4)+p(n-5)+\cdots-p(n)$
$(n-2,2)$	$p(n-2)+p(n-3)+3p(n-4)+3p(n-5)+5p(n-6)+\cdots-p(n-1)$
$(n-2,1^2)$	$p(n)+p(n-3)+p(n-4)+3p(n-5)+3p(n-6)+\cdots-p(n-1)-p(n-2)$
$(n-3,3)$	$p(n-3)+4p(n-5)+7p(n-6)+12p(n-7)+\cdots-2p(n-2)$
$(n-3,2,1)$	$p(n-1)+p(n-4)+5p(n-5)+10p(n-6)+\cdots-p(n-2)-2p(n-3)$
$(n-3,1^3)$	$p(n-1)+p(n-2)+p(n-4)+p(n-5)+6p(n-6)+\cdots-p(n)$

In each case, all remaining terms are positive and the terms shown are enough to establish the positivity of the row sum via Lemma 2 and its weaker consequence that

$$p(n-1) < p(n) < 2p(n-1) \text{ (for } n \geq 3\text{)}.$$

For example, $p(n-3)+4p(n-5)+7p(n-6)+12p(n-7) > p(n-3)+4p(n-5)+7p(n-5) = p(n-3)+11p(n-5) > p(n-3)+5p(n-4) > 2p(n-3)+3p(n-4) > 2p(n-2)$.

More generally, the generating function for a particular row sum will always be a rational function times the generating function for $p(n)$. The resulting row sum has finitely many subtracted terms because of the dominance of the part from $\rho = (1^k)$, $\sigma = \emptyset$, which contributes $\binom{a_1}{k}$ to the character polynomial. Hardy and Ramanujan showed that $p(n) \sim \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}$ [3], which is useful for proving the following result.

Theorem 1. *For a fixed $\lambda \vdash k$ and sufficiently large n, the row sum for $(n-k,\lambda)$ is positive.*

Proof. Because the row sum for a particular row in terms of the partition function has a fixed number of negative terms and an increasing number of positive terms as n increases, it suffices to show that $p(0)+p(1)+p(2)+\cdots+p(n-1) > cp(n)$ for any positive c and sufficiently large n. Let $f(n) = \sum_{i=0}^{n-1} p(i) - cp(n)$. It will suffice to show that $f(n) > f(n-1)$ for large values of n since a strictly increasing function with integer values must eventually be positive. After subtracting off common terms we are left to show that $p(n-1) - cp(n) > -cp(n-1)$, equivalently $\frac{p(n-1)}{p(n)} > \frac{c}{c+1}$.

We show that $\lim_{n\to\infty} \frac{p(n-1)}{p(n)} = 1$ from the asymptotic formula for $p(n)$, which implies that $\frac{p(n-1)}{p(n)} > \frac{c}{c+1}$ for n large enough since $\frac{c}{c+1} < 1$. To that end, $\frac{p(n-1)}{p(n)} \sim \frac{n}{n-1}e^{\pi\sqrt{2/3}(\sqrt{n-1}-\sqrt{n})} \to 1$ as $n \to \infty$. It would also suffice to use the weaker fact that while increasing, $p(n)$ grows slower than any exponential function. $\square$

4 Rows From The Bottom

The bottom row comes from the sign partition, where the characters are 1 if the cycle type has an even number of even parts and -1 for an odd number of even parts.

Proposition 1. *The sum of the bottom row of the character table for S_n is the number of self-conjugate partitions of n.*

Proof. It is well known that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts [5]. We construct a sign-reversing involution as follows. If a partition has a repeated odd part a appearing 2^k times such that $2^k a$ is greater than or equal to the largest even part, then replace the 2^k a's by one $2^k a$. If there is more than one such repeated odd part, then only make the greatest possible even part. Otherwise, if the partition has at least one even part, then replace the largest even part $2^k a$ (where a is odd) by 2^k a's. Since this map as an involution (it is its own inverse) and it changes the number of even parts by exactly one, the row sum is therefore the sum for the fixed points of this involution, which are partitions into distinct odd parts, each of which contributes $+1$ to the sum. □

The original goal of this research was to find a sign-reversing involution on decompositions of partitions into border strips via the Murnaghan-Nakayama rule [4] to prove the positivity of row sums. The proof of Proposition 1 is the only one found so far.

Let $s(n)$ denote the number of self-conjugate partitions of n, equivalently partitions of n into distinct odd parts. When two rows of the character table correspond to conjugate partitions, then the characters of one can be obtained from the other by multiplying by the sign representation. It follows from Proposition 1 that

$$\sum_{n=0}^{\infty} s(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1+(-1)^i x^i},$$

thus by repeating the proof of Lemma 1 with this generating function we get that every formula for a row sum in terms of $p(n)$ becomes the same formula for the conjugate row in terms of $s(n)$.

Lemma 3. *For $n \geq 2$, $s(n-1) \leq s(n) \leq s(n-1)+s(n-2)$. The second inequality is strict for $n \geq 9$ and the first for $n \geq 27$.*

Proof. If a partition of n into distinct odd parts contains a 1, then removing the 1 provides a unique partition of $n-1$ into distinct odd parts. If the partition does not contain a 1, then subtracting 2 from the least part yields a unique partition of $n-2$ into distinct odd parts. This provides a one-to-one map from partitions of n into distinct odd parts into those of $n-1$ and $n-2$, and it is not onto for $n \geq 9$ since we will not get $(n-2,1) \vdash n-1$ when n is odd and $n-2>1$ and we will not get $(n-5,3,1) \vdash n-1$ when n is even and $n-5>3$.

For the first inequality, consider a partition of $n-1$ into distinct odd parts. If it does not contain a 1, then we can append a 1 to get a partition of n. If it does contain a 1, then we can remove the 1 and increase the largest part by 2 to get a partition of n, thus providing a one-to-one map into partitions of n into distinct odd parts as long as $n-1>1$ so that we did not start with the partition (1). The strict inequality in this case comes from missing partitions that do not contain a 1 and whose largest two parts differ by 2. Examples can be constructed in four cases.
If $n \equiv 0 \pmod 4$ then $(m, m-2)$ is such a partition where $m \geq 5$ is odd and $n = 2m-2 \geq 8$.
If $n \equiv 1 \pmod 4$ then $(m, m-2, 5)$ is such a partition where $m \geq 9$ is odd and $n = 2m+3 \geq 21$.
If $n \equiv 2 \pmod 4$ then $(m, m-2, 7, 3)$ is such a partition where $m \geq 11$ is odd and $n = 2m+8 \geq 30$.
If $n \equiv 3 \pmod 4$ then $(m, m-2, 3)$ is such a partition where $m \geq 7$ is odd and $n = 2m+1 \geq 15$.
Considering these four cases, the largest value of n that does not have such a partition is 26, and indeed $s(25) = s(26) = 12$. □

Finally, since $s(n)$ is an increasing function bounded above by $p(n)$, we have $\lim_{n\to\infty} \frac{s(n-1)}{s(n)} = 1$. Putting it all together yields the following theorem.

Theorem 2. *The row sums corresponding to the partitions of n with largest part $\geq n-3$ and their conjugates are positive for all $n > 2$.*
For a fixed $\lambda \vdash k$ and sufficiently large n, the row sums for $(n-k, \lambda)$ and its conjugate are positive.

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