Chain lengths in the type B Tamari lattice

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Abstract

We find the largest union of two chains in the type B Tamari lattice by generalizing the techniques used for the classical (type A) Tamari lattice with a description of the type B case due to Hugh Thomas.

The Tamari lattice is a partially ordered set originally defined in terms of binary-operation bracketings [4]. Its elements are enumerated by Catalan numbers, so it is not surprising that there are several equivalent descriptions. One description comes from the Symmetric group in a way that can be generalized to other Coxeter groups, giving rise to Cambrian lattices [5]. Our focus is on the type B Tamari lattice that thus arises. A description of the elements of this poset due to Thomas [6] allows us to use similar techniques to those employed in classical Tamari lattice to find the largest union of two chains [1].

The elements of the Tamari lattice T_n can be represented by n-tuples $(v_1, \ldots, v_n)$ of integers from 1 to n following two rules [4]:
(i) each $v_i \geq i$
(ii) if $i \leq j \leq v_i$, then $v_j \leq v_i$.

The partial ordering is then componentwise comparison, so the technical complexity of the description of the elements yields a very simple ordering. For the purposes of this paper, the only thing we need to know about the type B Tamari lattice T_n^B is that its elements can be represented by n-tuples $(r_1, \ldots, r_n)$ of symbols from $\{0, 1, 2, \ldots, n-1, \infty\}$ following two rules [6]:
(i) for $i < j$, $r_i \leq r_j - (j - i)$ if $r_j - (j - i)$ is nonnegative
(ii) if $\infty > r_i \geq i$, then $r_{n+i-r_i} = \infty$.

While the description of the elements in this case is evidently more complicated, the ordering remains componentwise comparison. The bottom element of the poset will be $(0, \ldots, 0)$ and the top element will be $(\infty, \ldots, \infty)$. The first property implies that any time an entry $r_j = k$ for a finite number k, the prior k entries must also be finite and are at most $0, 1, \ldots, k-1$ in that order. In particular, consecutive entries can only be equal if they are both 0 or both ∞, and an ∞ can only be followed by 0 or ∞. The second property says that large entries in the n-tuple force infinities elsewhere. For example, anything that starts with 1 must end with ∞. The largest element with all finite entries is $(0, 1, \ldots, n-1)$. Hasse diagrams for T_2^B and T_3^B are shown in Figure 1.

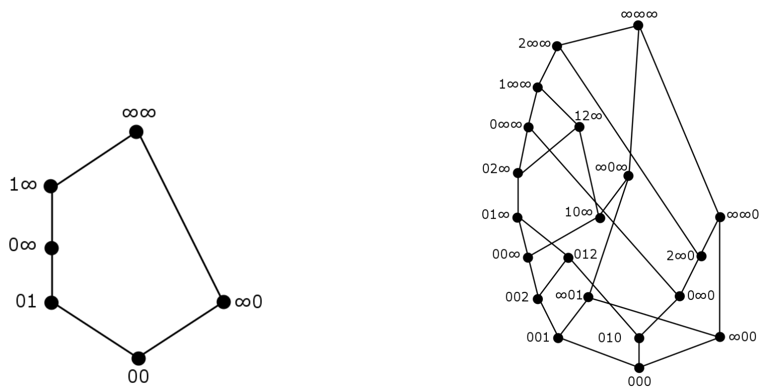

Figure 1: T_2^B and T_3^B

Given any poset P, there exists a partition $\lambda(P)$ such that the sum of the first k parts of $\lambda(P)$ is the maximum number of elements in a union of k chains in P. A theorem of Greene and Kleitman states that the conjugate of $\lambda(P)$ has the same property for antichains in P [2, 3]. Let $\lambda_k(P)$ denote the kth part of this partition.

The poset T_n^B is not graded, so the key to constructing two disjoint chains of maximal length is to extract the "leveled" elements. A poset is *leveled* if every element (except those on the top and bottom levels) covers something on the level below and is covered by something on the level above. Every element in a leveled poset P is on one of the longest chains of P, but this is a slightly weaker condition than being a graded poset where every maximal chain must have the same length.

Consider the subposet of elements of T_n^B that are on a chain of maximal length. For such an element μ, define $r(\mu)$ to be the length of the longest chain from the bottom element to μ.

Lemma 1. *Leveled elements appear on level $r(\mu)$, which is the sum of the entries of μ counting each ∞ as n. Unleveled elements drawn at the lowest possible level will be at or below the level of their sum.*

Proof. Among the leveled elements we can construct at least one chain where each step increases only one entry by exactly 1. An explicit description of such a chain will be given in the proof of Theorem 1. For the unleveled elements, every covering relation must involve increasing at least one part by at least 1, so the sum provides an upper bound on the maximum number of coverings needed to reach that element from the bottom. □ □

The longest chain therefore has length n^2, so $\lambda_1(T_n^B) = n^2 + 1$. Our main result is the following.

Theorem 1. *For $n \geq 4$, $\lambda_2(T_n^B) = \lambda_1(T_n^B) - 5$.*

Proof. First we construct two disjoint chains of the desired lengths, then show that no larger union is possible. The first chain starts at $(0, \ldots, 0)$ and increases the rightmost possible part by 1, except that adding 1 to $n-1$ yields ∞. For example, when $n = 4$ the chain is (0,0,0,0), (0,0,0,1), (0,0,0,2), (0,0,0,3), (0,0,0,∞), (0,0,1,∞), (0,0,2,∞), (0,0,3,∞), (0,0,∞,∞), (0,1,∞,∞), (0,2,∞,∞), (0,3,∞,∞), (0,∞,∞,∞), (1,∞,∞,∞), (2,∞,∞,∞), (3,∞,∞,∞), (∞,∞,∞,∞).

The second chain begins at $(0, \dots, 0, 1, 2)$ and increases the leftmost possible part by 1, ending at $(n-2, n-1, \infty, \dots, \infty)$. When $n = 4$, the chain is (0,0,1,2), (0,0,1,3), (0,0,2,3), (0,1,2,3), (0,1,2,∞), (0,1,3,∞), (0,2,3,∞), (1,2,3,∞), (1,2,∞,∞), (1,3,∞,∞), (2,3,∞,∞).

These two chains are disjoint because for larger values of n the elements that are on the same level from each chain will have a different number of 0 and/or ∞ entries. In particular, the first chain gains an ∞ on every nth level. Up to level n the first chain has only one non-zero entry while the second chain always has at least two nonzero entries. The second chain acquires its first ∞ at level $\frac{n(n-1)}{2} + 1$, which is greater than $2n$ for $n \geq 5$ and has only one 0 at that point, which becomes a 1 $n-1$ levels later. The first chain has at least one 0 until level $n^2 - n + 1$. At that point all but one entry is ∞, while the second chain always has at least two finite entries, thus establishing that the two chains are disjoint.

This second chain is six elements smaller than the first rather than the five needed for the theorem. We append the unleveled element $(0, \dots, 0, 1, 0)$ at the beginning of the second chain to make it one longer, constructively proving that $\lambda_2(T_n^B) \geq \lambda_1(T_n^B) - 5$.

To show that two chains cannot contain any more elements it suffices (by the Greene-Kleitman theorem) to show that T_n^B can be partitioned into $n^2 + 1$ antichains, five of which comprise a single element. To that end, we start by observing that a Hasse diagram provides a decomposition into antichains via its levels. To get the desired antichains, start with every element placed on the lowest possible level. All of the unleveled elements can be shifted up one level as long as they are all moved together. This makes the bottom two levels comprise one element each, namely $(0, \dots, 0)$ and $(0, \dots, 0, 1)$. The top element $(\infty, \dots, \infty)$ covers elements with only one finite entry. One of these is the leveled element $(n-1, \infty, \dots, \infty)$ and the rest contain a 0 because no other finite number can be preceded by ∞. The unleveled elements covered by the top element are therefore started at least n levels down by Lemma 1, so even moving them up 1 still keeps them out of the top three levels for $n \geq 4$. The elements covered by $(n-1, \infty, \dots, \infty)$ are similarly the leveled element $(n-2, \infty, \dots, \infty)$ and those that replace an ∞ with a 0 and thus start too many levels down to interfere. We therefore have the antichains to show that $\lambda_2(T_n^B) \leq \lambda_1(T_n^B) - 5$, which together with the chain construction completes the proof. □

An unlabeled Hasse diagram of T_4^B is shown in Figure 2. The only leveled elements in this poset are the ones used in the construction.

Unlike T_n, T_n^B is not self-dual, hence the odd difference in chain lengths, though the leveled elements are self-dual. A third long disjoint chain begins to emerge at $n = 5$, where there are six levels of one element each and four levels of two elements each within the leveled elements. It appears likely that $\lambda_k(T_n^B) - \lambda_{k+1}(T_n^B)$ will be constant for fixed k and large n, so computing that value for $k > 1$ is a natural followup problem. Another area exploration is the type D analogue.

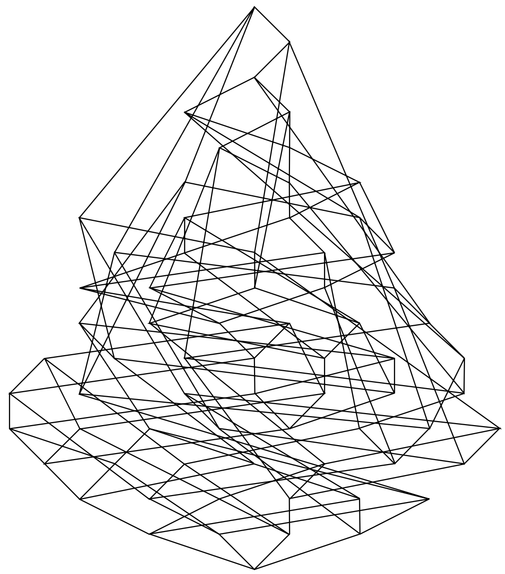

Figure 2: T_4^B

Acknowledgements

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